State-space modeling using first-principles

Lecture 2

Principles of Modeling for Cyber-Physical Systems

Instructor: Madhur Behl

Download Matlab

Campus-wide license for MATLAB, Simulink, and companion toolboxes

https://www.mathworks.com/academia/tah-portal/university-of-virginia-40704757.html
(or search for UVA Matlab portal)

Contact res-consult@virginia.edu for questions regarding access to Matlab licenses.

In today's lecture we will learn about…

> Prediction is very difficult, especially if it's about the future.
>
> — Niels Bohr —

Madhur Behl - madhur.behl@virginia.edu

In today's lecture we will learn about…

How to predict the future states and outputs of systems using physics based mathematical modeling

In today's lecture we will learn about…

- Ordinary differential equations (ODEs).
- Linear dynamical systems
- State-space representation
- Elements of first-principles based modeling:
  - Mechanical and electrical modeling

What is a System ?

Cruise control system

Cardio-pulmonary system

Autopilot system

Economic system

Governance system

Grading system

Tropical storm system

Communication system

Complex system

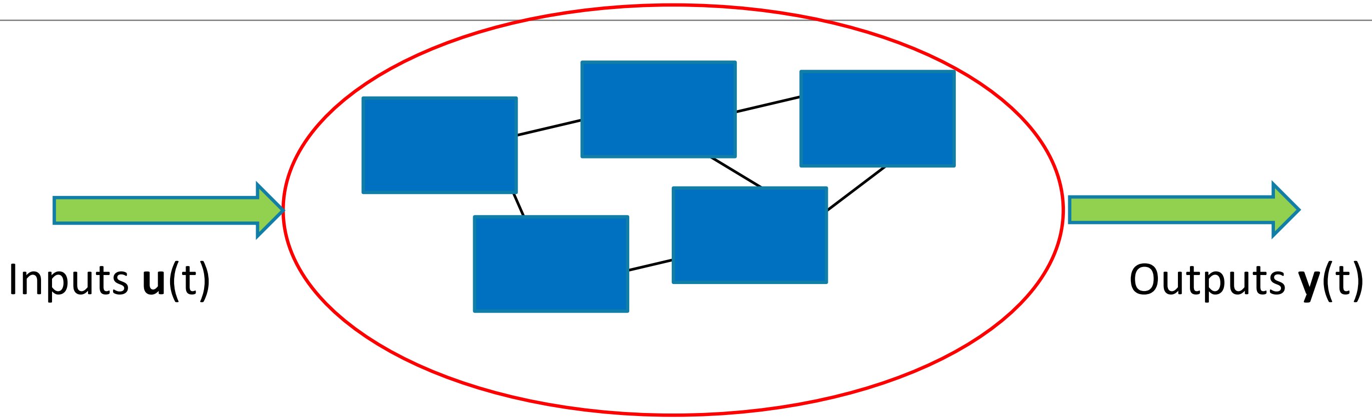
System of systems

Taxation system

Cyber-Physical systems

Healthcare system

What is a System ? Intuitive defintion

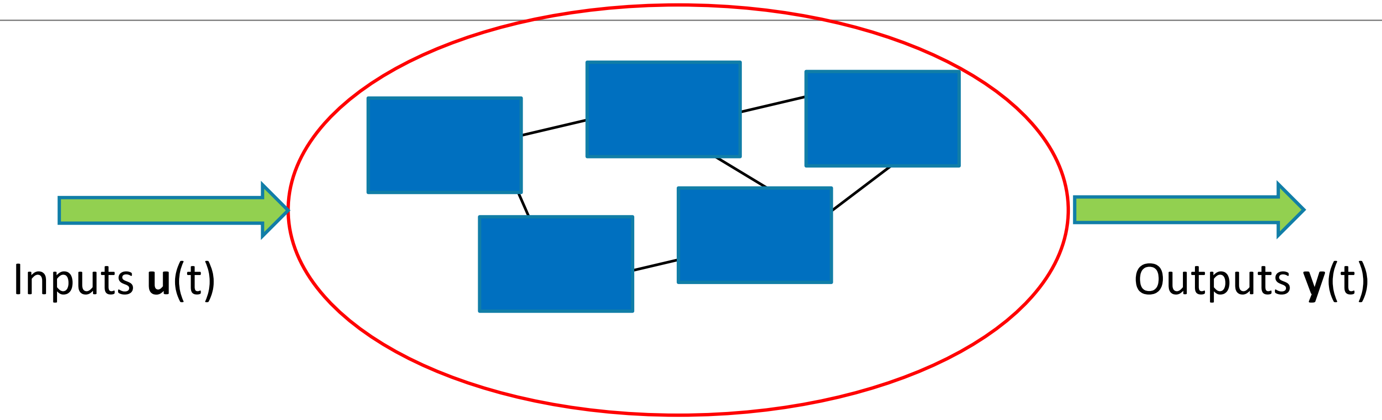


Collection of components

Non-trivial interactions

Well defined boundary
with the environment

What is a System ?



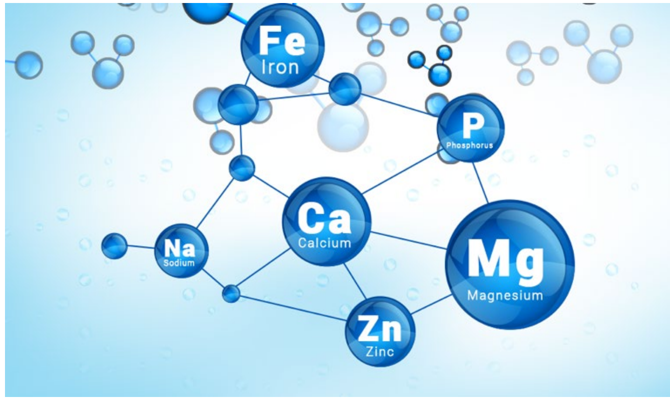
Mapping from time dependent inputs to time dependent outputs

(causal definition)

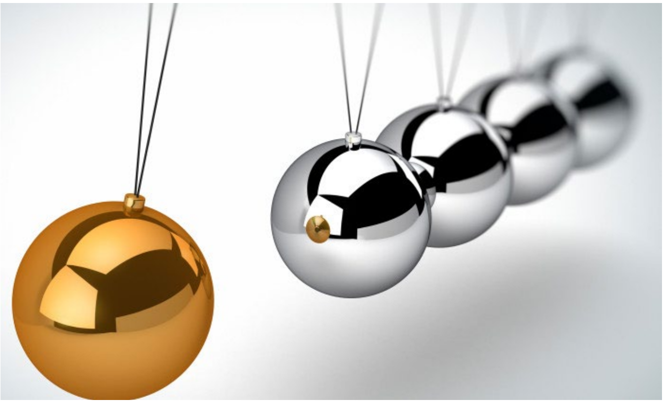
Differential equations

Many phenomena can be expressed by equations which involve the **rates of change** of quantities (position, population, concentraition, temperature…) that describe the **state** of the phenomena.

Economics



Chemistry



Mechanics

Engineering

Social Science

Biology

Madhur Behl - madhur.behl@virginia.edu

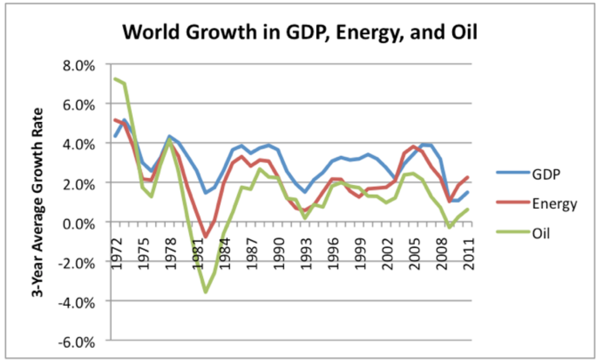
The *state* of a system describes enough information about the system to determine its future behavior in the absence of any external inputs affecting the system.

The set of possible combinations of state variable values is called the **state space** of the system.

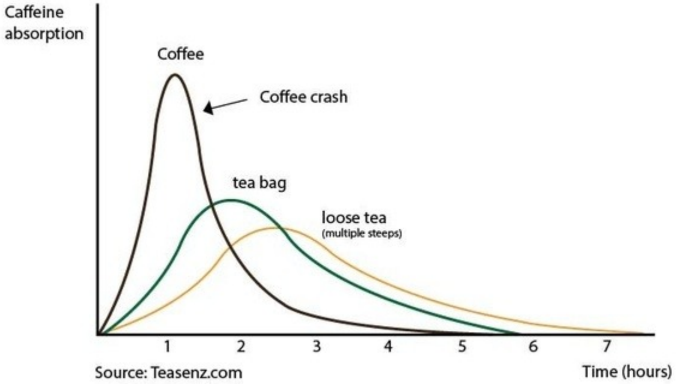
Differential equations

The state of the system is characterized by **state variables**, which describe the system.

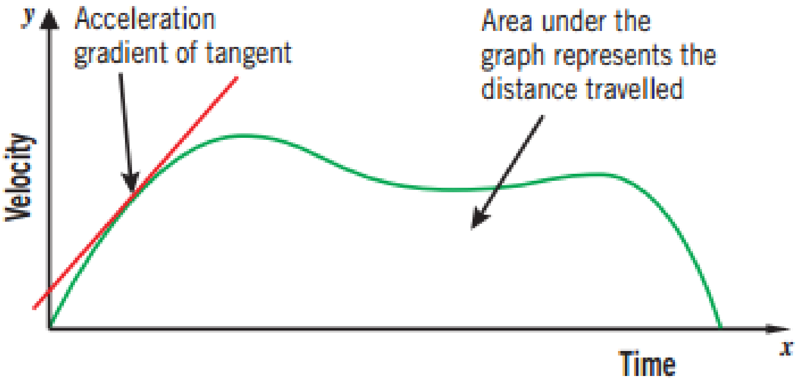
The rate of change is (usually) expressed with respect to time



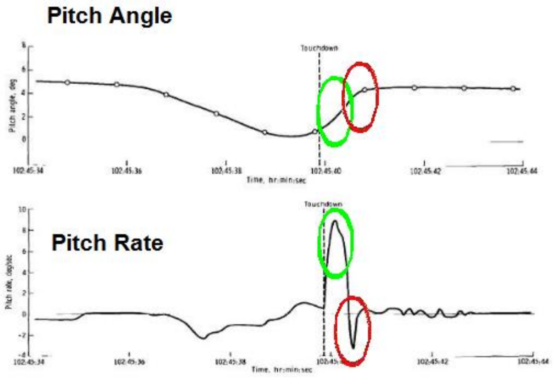
Economics



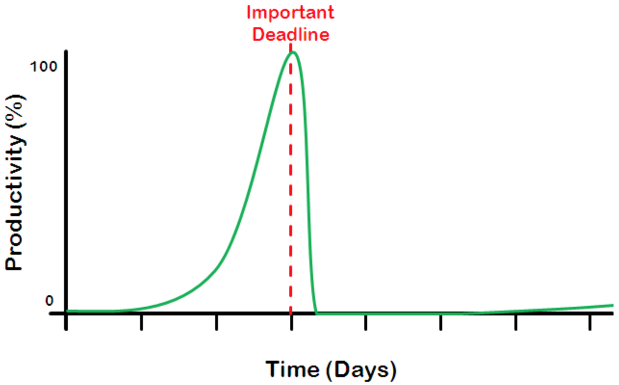
Chemistry



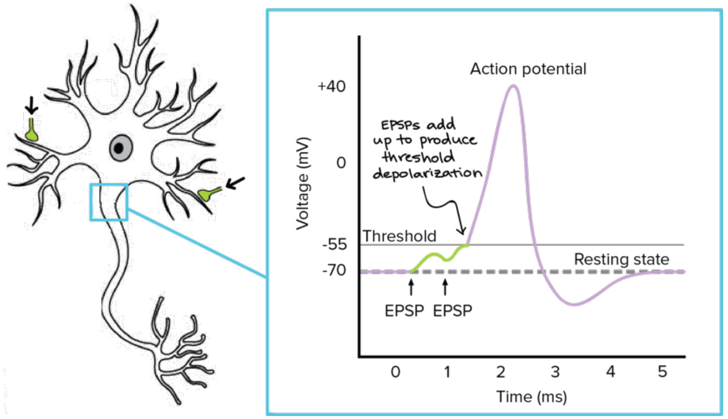
Mechanics



Engineering



Social Science



Biology

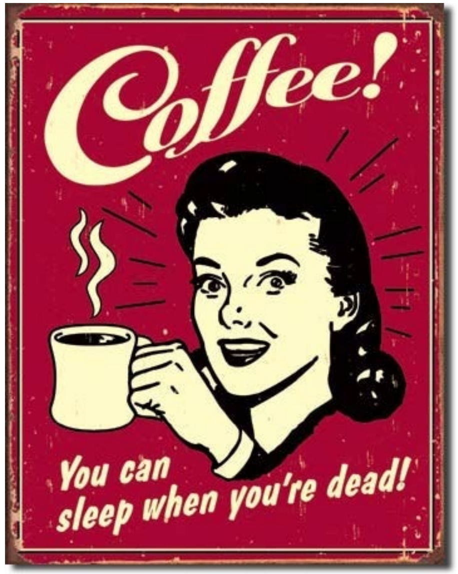
Differential equations – A simple example

After drinking a cup of coffee, the amount C of caffeine in person's body follows the differential equation:

$$\frac{dC}{dt} = -\alpha C$$

1st order

Where the constant α has a value of 0.14 hour^{-1}



How many hours will it take to metabolize half of the initial amount of caffeine ?

$$\int \frac{dC}{C} = -\alpha \int dt \quad ; \quad C(t) = C_0 e^{-\alpha t} \; ; \; if\ C(t) = C_0/2, \; t = ln2/\alpha$$

Differential equations –example

- Susceptibles S_t
- Infectious I_t
- Recovered or dead R_t

DOI: 10.1007/978-1-4757-3516-1 • Corpus ID: 83264573

Mathematical Models in Population Biology and Epidemiology

F. Brauer, C. Castillo-Chavez • Published 2001 • Biology

$$S'(t) = -\beta S(t)I(t), \qquad I'(t) = \beta S(t)I(t) - \gamma I(t), \qquad R'(t) = \gamma I(t),$$

$$\frac{dS(t)}{dt}$$

$$S(t) + I(t) + R(t) = 1$$

Recall: Differential equations

- Ordinary differential equation (ODE): all derivatives are with respect to single independent variable, often representing time.
- Order of ODE is determined by highest-order derivative of state variable function appearing in ODE.
- ODE with higher-order derivatives can be transformed into equivalent first-order system.
- Most ODE software's are designed to solve only first-order equations.

Higher order ODE's

For k-th order ODE

$$y^{(k)}(t) = f(t, y, y', \ldots, y^{(k-1)})$$

define k new unknown functions

$$u_1(t) = y(t),\ u_2(t) = y'(t),\ \ldots,\ u_k(t) = y^{(k-1)}(t)$$

Then original ODE is equivalent to first-order system

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{k-1}'(t) \\ u_k'(t) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ u_3(t) \\ \vdots \\ u_k(t) \\ f(t, u_1, u_2, \ldots, u_k) \end{bmatrix}$$

What makes a system dynamic ?

Inputs change with time ?

Outputs change with time ?

USD

$100 

$200 

$300 

Currency Exchange System

Euro

€85 

€170 

€255 

Static vs Dynamic Systems

Static System

Output is determined only by the current input, reacts instantaneously

Relationship between the inputs and outputs does not change (it is static!)

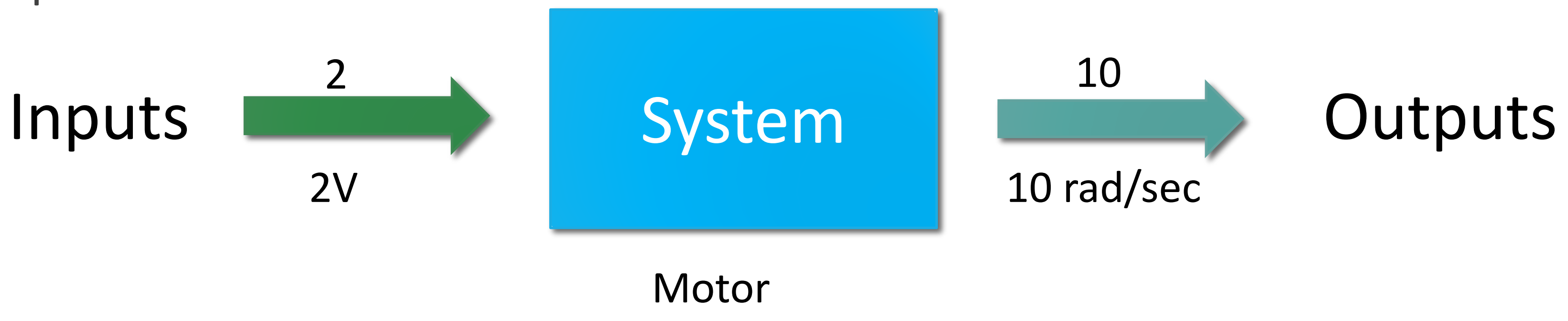
Relationship is represented by an algebraic equation

Dynamic System

Output takes time to react

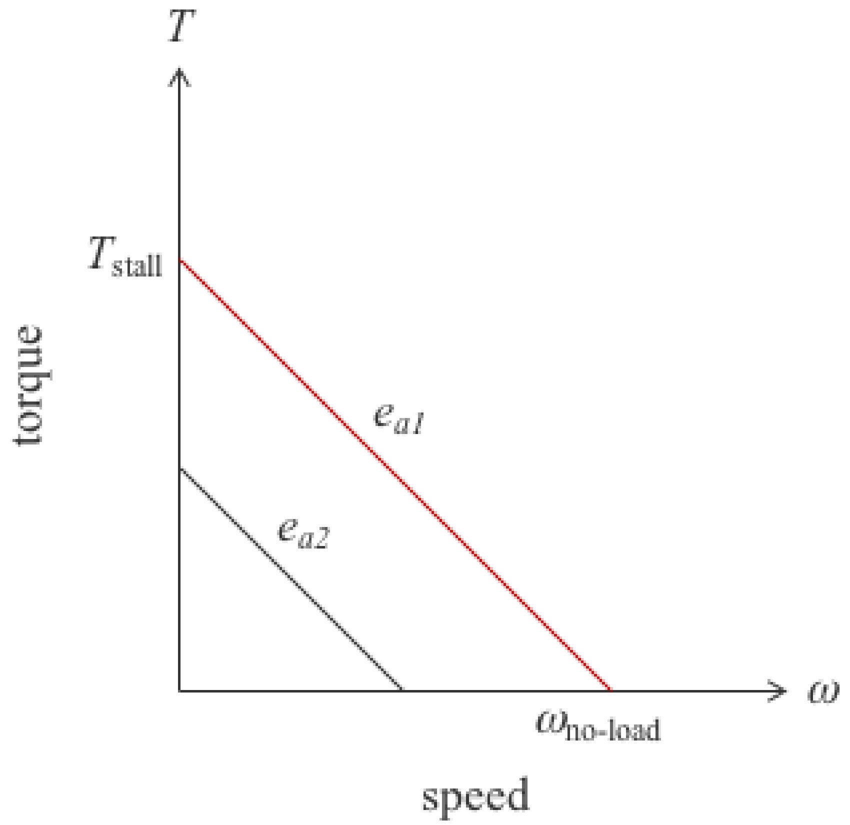
Relationship changes with time, depends on past inputs and initial conditions (it is dynamic!)

Relationship is represented by a differential equation

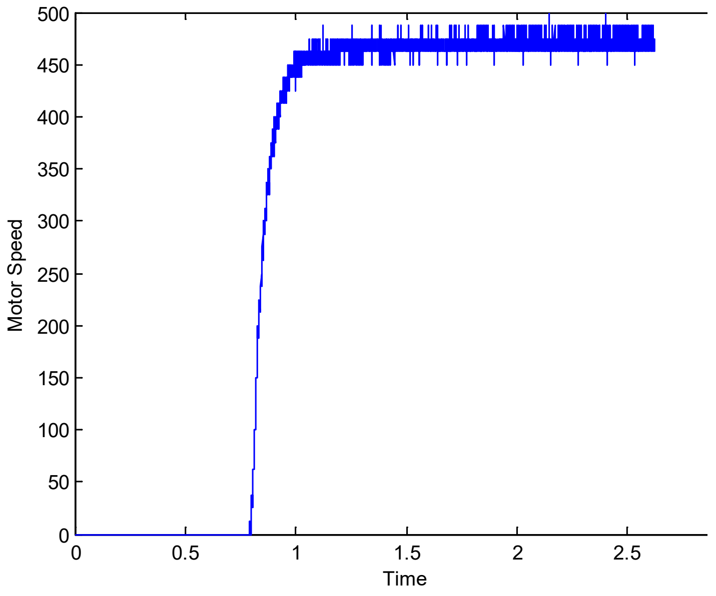


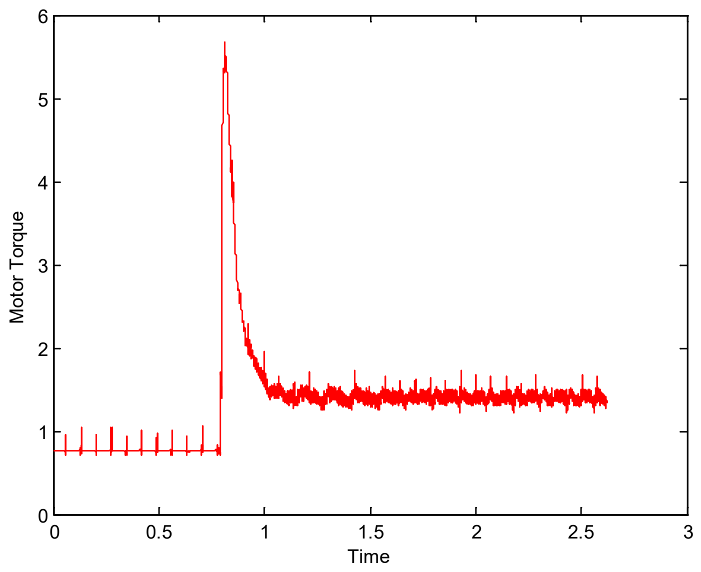
Static vs Dynamic Systems

Static System viewpoint



Dynamic System viewpoint





Dynamical System

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Dynamical System

Possibly a non-linear function

$$\frac{dx}{dt} = \dot{x} = \underbrace{f(x(t), u(t), t)}$$

Rate of change

The state $x(t_1)$ at any future time, may be determined exactly given knowledge of the initial state, $x(t_0)$ and the time history of the inputs, $u(t)$ between t_0 and t_1

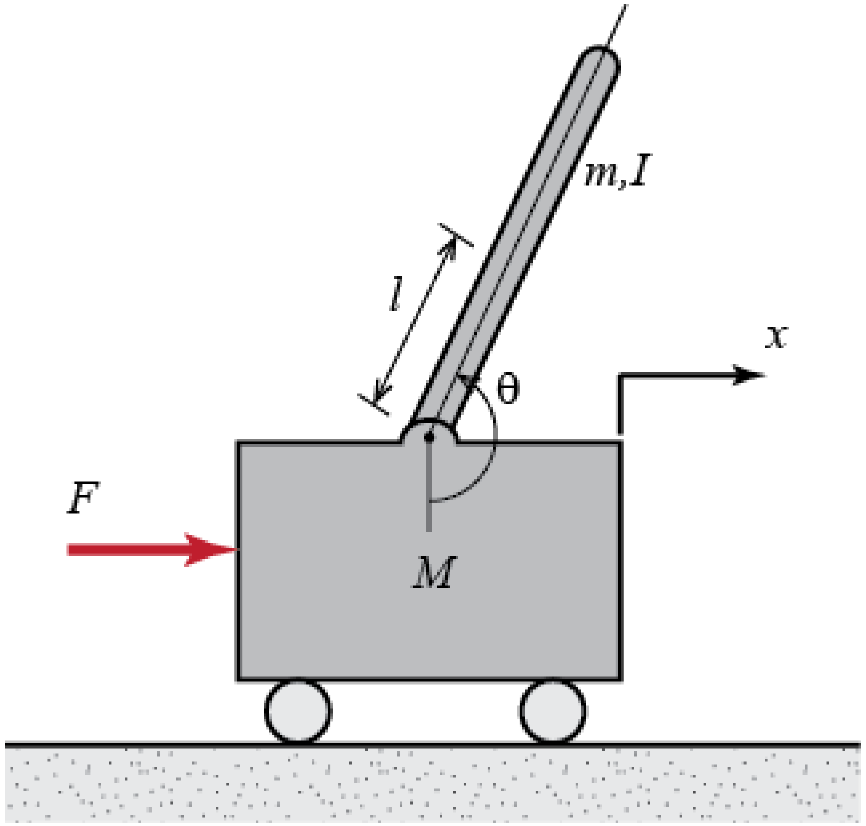
System order: n, min number of states required for the above statement to be true.

Possibly a non-linear function

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Rate of change

Inverted pendulum

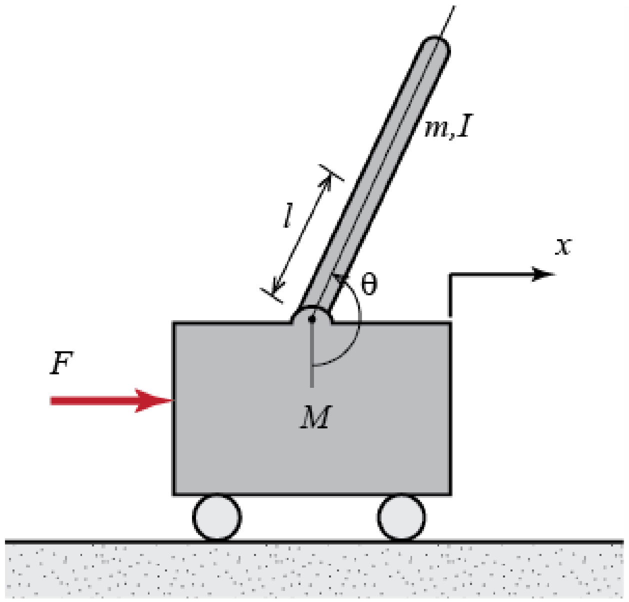


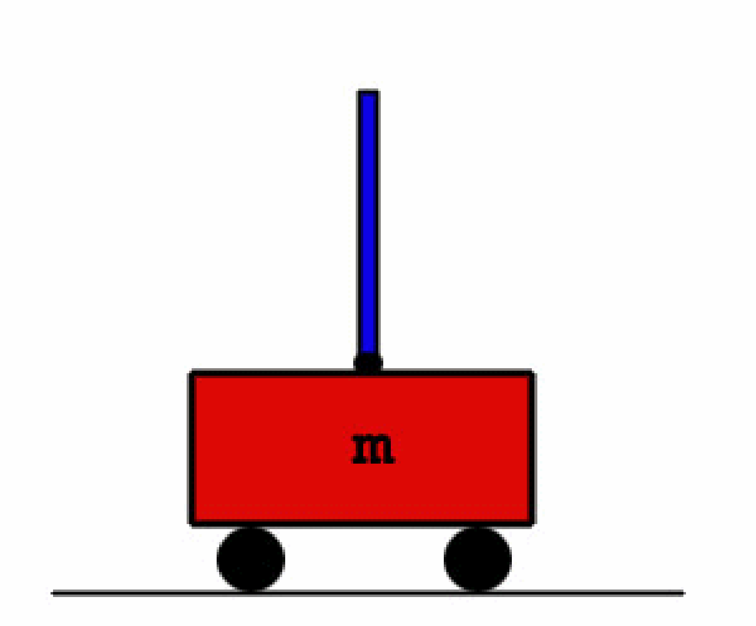
- Inverted pendulum mounted to a motorized cart.
- Unstable without control :
  - pendulum will simply fall over if the cart isn't moved to balance it.

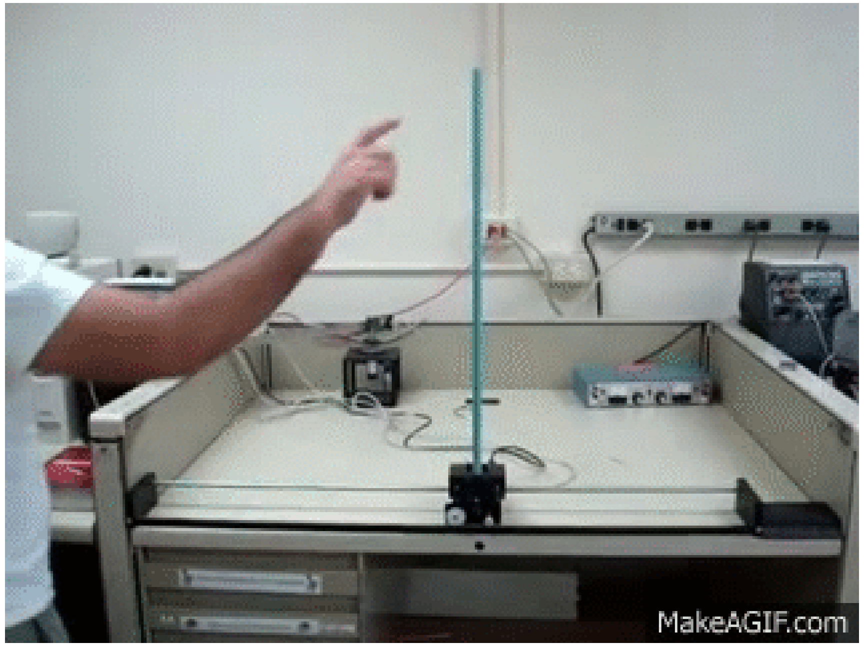
Balance the inverted pendulum by applying a force to the cart on which the pendulum is attached.

Inverted pendulum

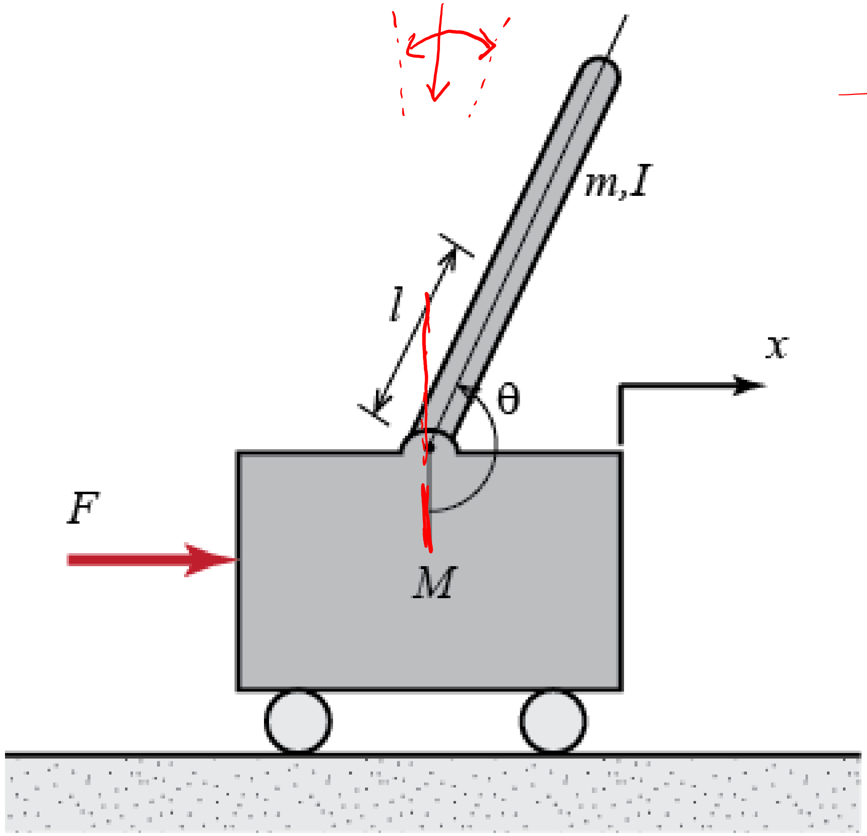
Inverted pendulum





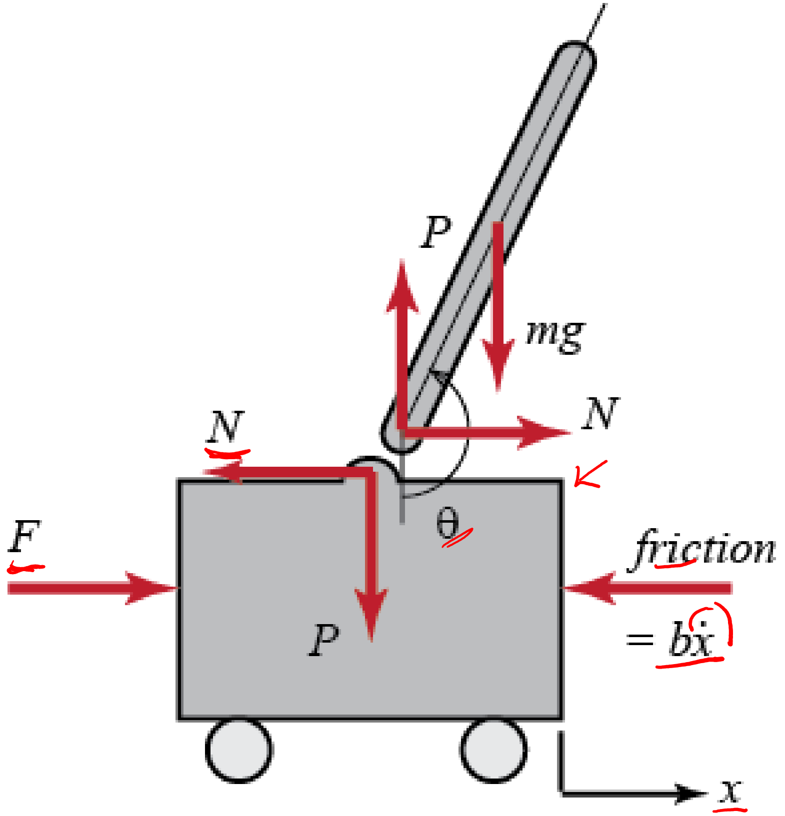


Inverted pendulum



- Initially pendulum begins with $\theta = \pi$
- Requirements:
  - Settling time for θ less than 5 secs.
  - Pendulum angle θ never exceeds 0.05 radians from the vertical.

Inverted pendulum – ODEs



Forces in the horizontal direction $M\ddot{x} + b\dot{x} + N = F$

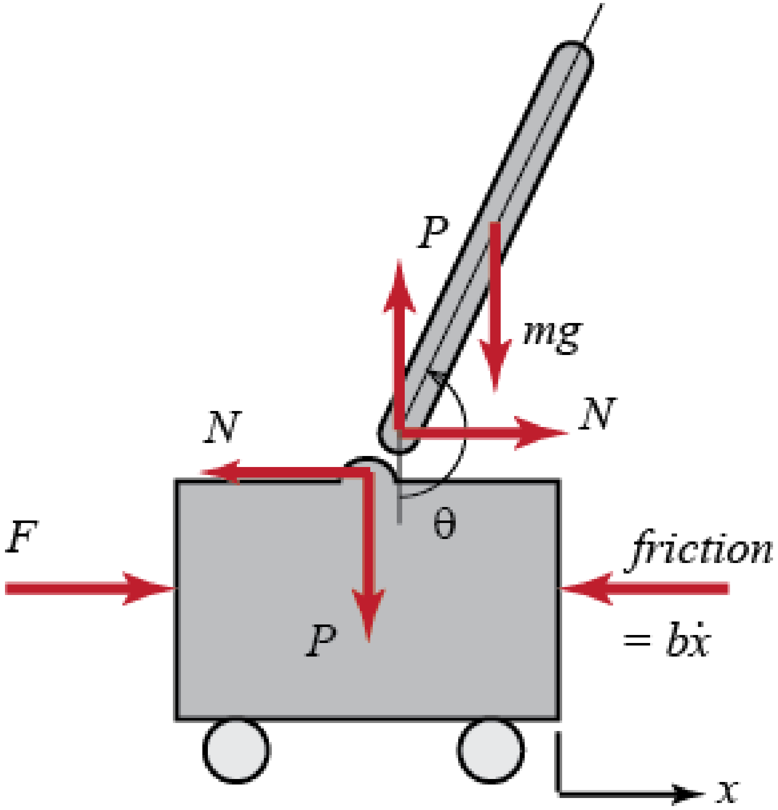
Reaction force N: $N = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml^2\dot{\theta}^2\sin\theta$

Governing equation (1) of this system: Horizontal

$$(M + m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml^2\dot{\theta}^2\sin\theta = F$$

1) ODE? ✓
2) 2nd od. ODE in x, θ
3) L / nL wrt x θ =

Inverted pendulum - – ODEs



Forces in the vertical direction:

$$P\sin\theta + N\cos\theta - mg\sin\theta = ml\ddot{\theta} + m\ddot{x}\cos\theta$$

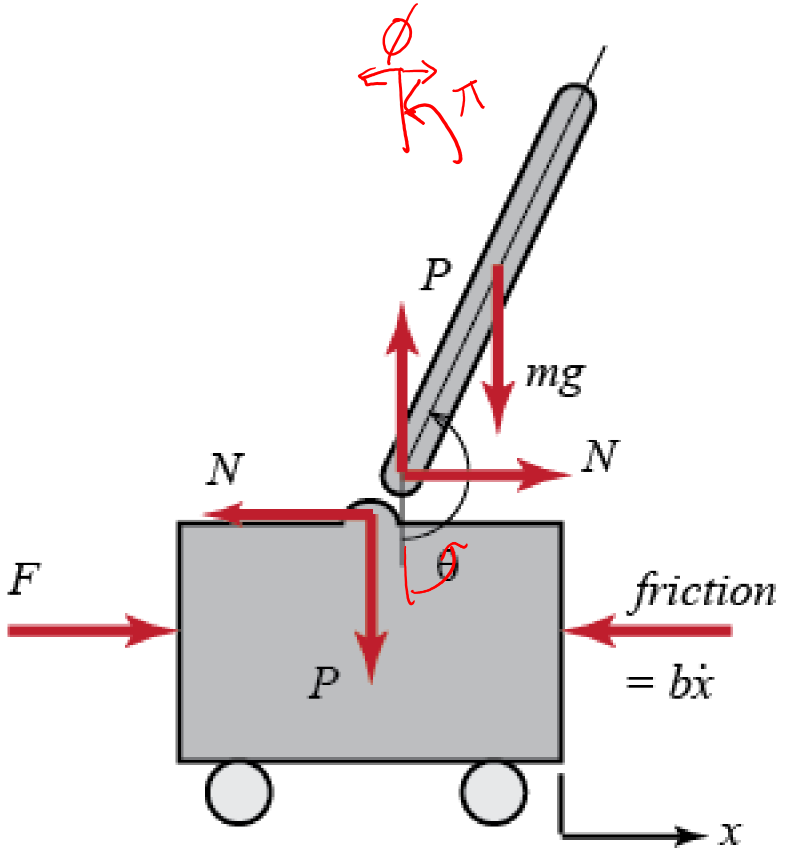
Get rid of the P and the N terms:
(moment balance equation)

$$-Pl\sin\theta - Nl\cos\theta = I\ddot{\theta}$$

Governing equation (2) of this system: Vertical

$$(I + ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$$

Inverted pendulum



Assuming that the system remains within a small neighborhood of the equilibrium $\theta = \pi$

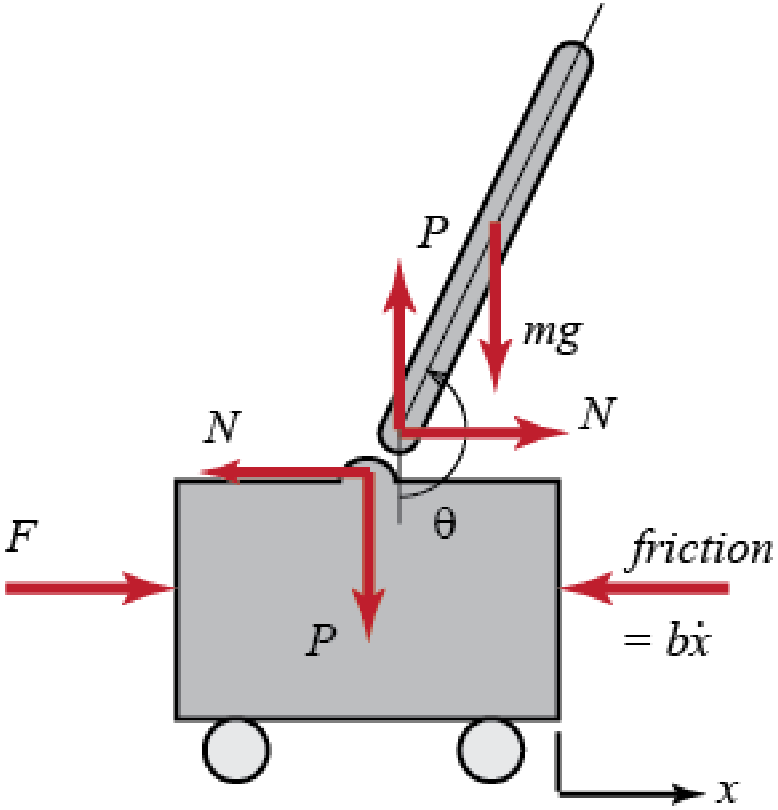
For small deviations $\emptyset$:

$$\cos(\pi + \emptyset) \approx -1$$

$$\sin(\pi + \emptyset) \approx -\emptyset$$

$$\dot{\theta}^2 = \emptyset^2 \approx 0$$

Inverted pendulum - Dynamics



Equations of motion are:

→2 1st ODE

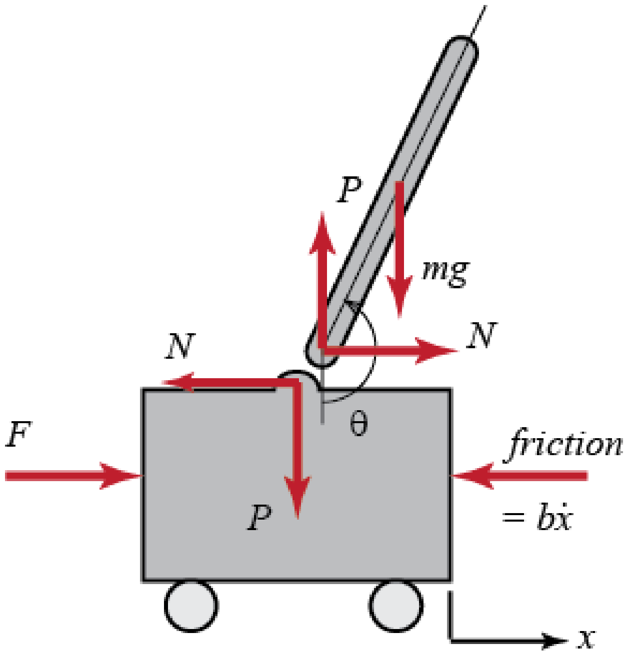
4

→2nd $(I + ml^2)\ddot{\phi} + mgl\phi = ml\ddot{x}$ (Linear)

→2nd $(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = F$ (L)

2 1OD ODE

Rearranging – State-Space representation



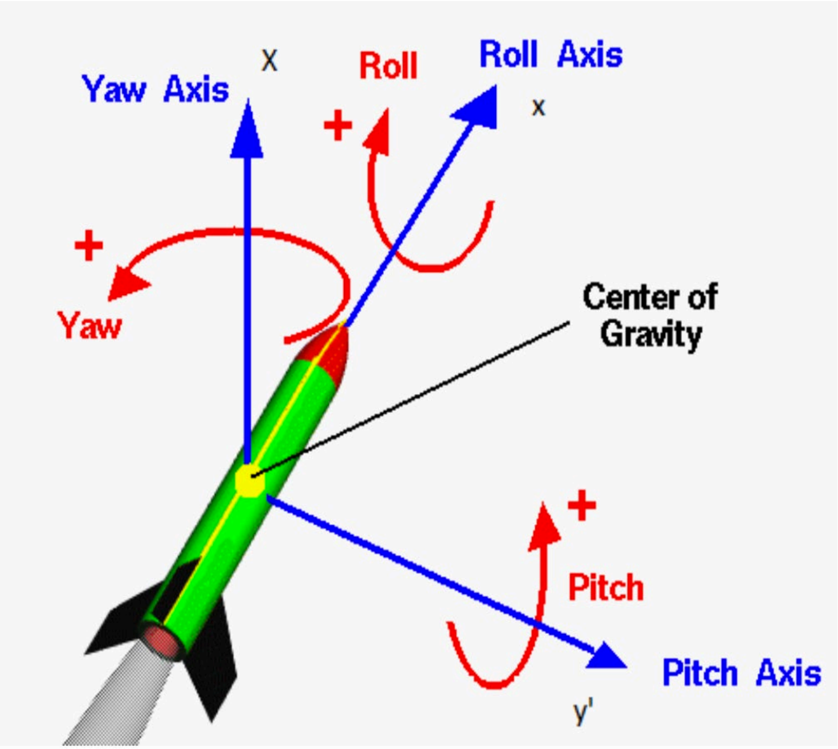
$\dot{X} = A X + B \cdot u$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} u$$

$y = C X + D \cdot u$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

From State-Space to Space..and back



$$\ddot{x} = \frac{1}{m}(F_x c\psi c\theta + F_y(c\psi s\theta s\phi - s\psi c\phi) + F_z(s\psi s\phi + c\psi s\theta c\phi)) - g$$
$$\ddot{y} = \frac{1}{m}(F_x s\psi c\theta + F_y(c\psi c\phi + s\psi s\theta s\phi) + F_z(s\psi s\theta c\phi - c\psi s\phi))$$
$$\ddot{z} = \frac{1}{m}(-F_x s\theta + F_y c\theta s\phi + F_z c\theta c\phi)$$

$$\ddot{\phi} = \frac{M_x}{I_a} + \dot{\psi}\dot{\theta}c\theta + \frac{s\theta}{I_t c\theta}\left(M_z c\phi + M_y s\phi + I_a\left(\dot{\phi}\dot{\theta} - \dot{\psi}\dot{\theta}s\theta\right) + 2I_t\dot{\psi}\dot{\theta}s\theta\right)$$
$$\ddot{\theta} = \frac{1}{I_t}(0.5(I_a - I_t)\dot{\psi}^2 s2\theta - I_a\dot{\phi}\dot{\psi}c\theta + M_y c\phi - M_z s\phi)$$
$$\ddot{\psi} = \frac{1}{I_t c\theta}\left(M_z c\phi + M_y s\phi + I_a(\dot{\phi}\dot{\theta} - \dot{\psi}\dot{\theta}s\theta) + 2I_t\dot{\psi}\dot{\theta}s\theta\right)$$

From state-space to Space

Dynamical System

Possibly a non-linear function

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Rate of change

Time invariant system: Simplifying assumption #1

f does not depend on time

m, m, l, b

$$\frac{dx}{dt} = \dot{x} = f(x, u)$$

u(t)

Rate of change

- The underlying physical laws themselves do not typically depend on time.
- Inputs u(t) may be time dependent
- The parameters/constants which describe the function f remain the same.

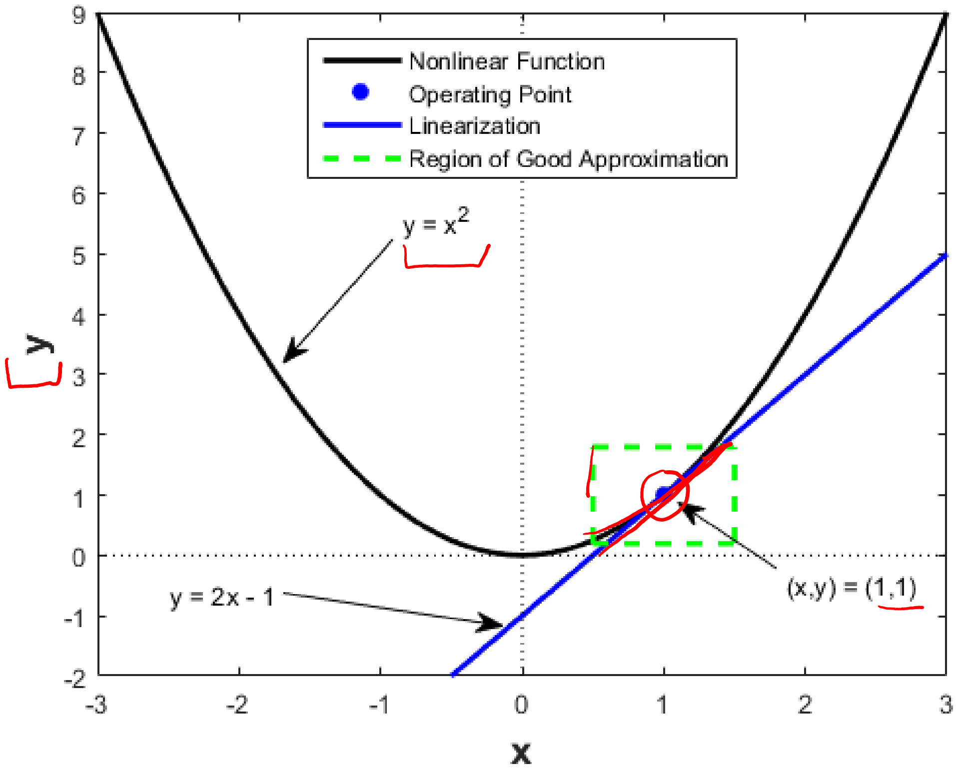
Madhur Behl - madhur.behl@virginia.edu

Linearity: Simplifying assumption #2

cos(π + ϕ) ≈ -1

Over a sufficiently small operating range (think tangent line near a curve), the dynamics of most systems are approximately **linear**

$$\dot{x} = Ax + Bu$$



State-Space representation

A state-space model represents a system by a series of first-order differential state equations and algebraic output equations.

Differential equations have been rearranged as a series of first order differential equations.

Example

Consider the following system where $u(t)$ is the input and $\dot{x}(t)$ is the output.

$$\dddot{x} + 5\ddot{x} + 3\dot{x} + 2x = u \, , y = \dot{x}$$

Can create a state-space model by pure mathematical manipulation through changing variables

$$x_1 = x, \; x_2 = \dot{x}, \; x_3 = \ddot{x}$$

Resulting in the following three first order differential equations (ODEs)

$$\dot{x_1} = x_2,$$

$$\dot{x_2} = x_3,$$

$$\dot{x_3} = -5x_3 - 3x_2 - 2x_1 + u$$

State Equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -5x_3 - 3x_2 - 2x_1 + u$$

Output Equation

$$y = x_2$$

System has 1 input (u), 1 output (y), and 3 state variables (x_1, x_2, x_3)

State-space representation

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$y = C\vec{x} + D\vec{u}$$

for linear systems

From our prior example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -5x_3 - 3x_2 - 2x_1 + u$$

$$y = x_2$$

$$\dot{X} = A X + B \cdot u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$-2x_1 \quad -3x_2 \quad -5x_3 \quad + \quad u$$

$$y = C \cdot X + D \cdot u$$

$$[y] = [\,0 \quad 1 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\,u$$

The State-Space Modeling Process

$u(t)$: u

1) Identify ***input*** variables (actuators and exogenous inputs).
2) Identify ***output*** variables (sensors and performance variables). y
3) Identify ***state*** variables. (Hmmm…how ? – indep. energy storage) x?
4) Use first principles of physics to relate derivative of state variables to the input, state, and the output variables.

Madhur Behl - madhur.behl@virginia.edu

Why use state-space representations ?

State-space models:

- are numerically efficient to solve,
- can handle complex systems,
- allow for a more geometric understanding of dynamic systems, and
- form the basis for much of modern control theory

Linear dynamical system

Continuous-time linear dynamical system (CT LDS) has the form

$$\dot{x} = A(t)x(t) + B(t)u(t) \qquad y(t) = C(t)x(t) + D(t)u(t)$$

- $t \in \mathbf{R}$ denotes *time*
- $x(t) \in \mathbf{R}^n$ is the *state* (vector)
- $u(t) \in \mathbf{R}^m$ is the *input* or *control*
- $y(t) \in \mathbf{R}^p$ is the *output*

Continuous-time linear dynamical system (CT LDS)

$$\dot{x} = A(t)x(t) + B(t)u(t) \qquad y(t) = C(t)x(t) + D(t)u(t)$$

x

- $A(t) \in \mathbf{R}^{n \times n}$ is the *dynamics matrix*
- $B(t) \in \mathbf{R}^{n \times m}$ is the *input matrix*
- $C(t) \in \mathbf{R}^{p \times n}$ is the *output* or *sensor matrix*
- $D(t) \in \mathbf{R}^{p \times m}$ is the *feedthrough matrix*

Linear dynamical system

Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, $i.e.$, don't depend on t

Linear dynamical system

Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, $i.e.$, don't depend on t

- when there is no input u (hence, no B or D) system is called *autonomous*

Linear dynamical system
Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, $i.e.$, don't depend on t

- when there is no input u (hence, no B or D) system is called *autonomous*

- very often there is no feedthrough, $i.e.$, $D = 0$

Madhur Behl - madhur.behl@virginia.edu

Linear dynamical system

Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, $i.e.$, don't depend on t

- when there is no input u (hence, no B or D) system is called *autonomous*

- very often there is no feedthrough, $i.e.$, $D = 0$

- when $u(t)$ and $y(t)$ are scalar, system is called *single-input, single-output* (SISO); when input & output signal dimensions are more than one, MIMO

Madhur Behl - madhur.behl@virginia.edu

Discrete-time (linear dynamical system) (DT LDS)

step size (sample time)

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

$$y(k) = C(k)x(k) + D(k)u(k)$$

where

- $k \in \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$
- (vector) signals x, u, y are *sequences*

Many dynamical systems are nonlinear (a fascinating topic) so why study linear systems?

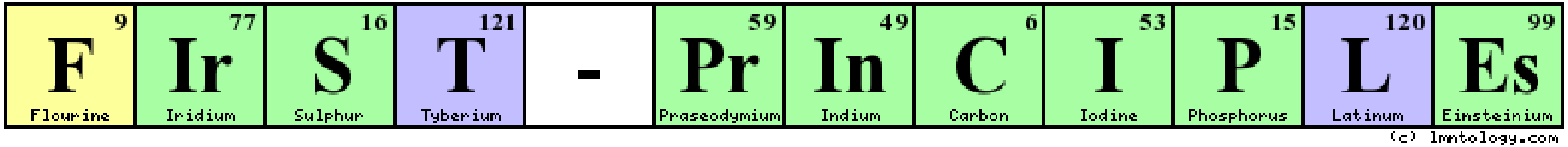
- Most techniques for nonlinear systems are based on linear systems.
- Methods for linear systems often work unreasonably well, in practice, for nonlinear systems.
- If you do not understand linear dynamical systems, you certainly cannot understand nonlinear dynamical systems.

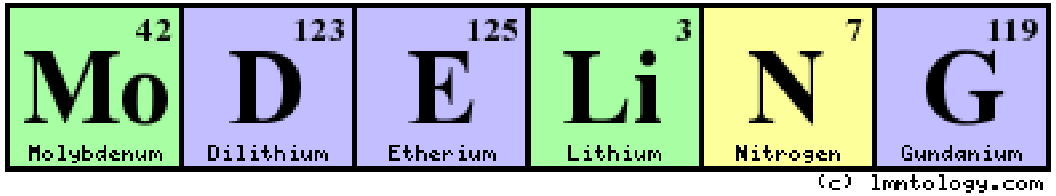
Many dynamical systems are nonlinear (a fascinating topic) so why study linear systems?

"Finally, we make some remarks on why linear systems are so important. The answer is simple: because **we can solve them!** "

- Richard Feynman [Fey63, p. 25-4]

Elements of..

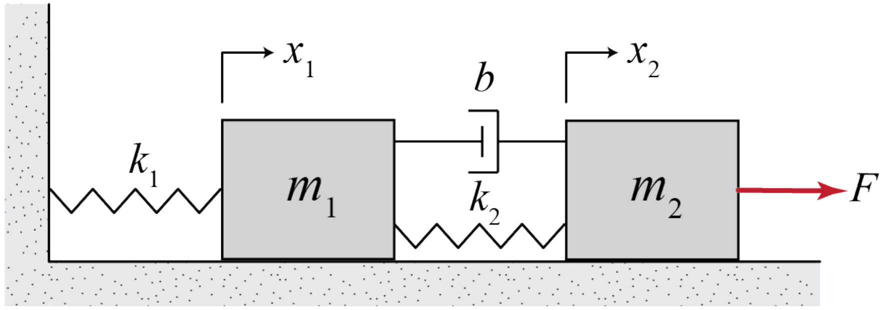

9 F Flourine
77 Ir Iridium
16 S Sulphur
121 T Tyberium
-
59 Pr Praseodymium
49 In Indium
6 C Carbon
53 I Iodine
15 P Phosphorus
120 L Latinum
99 Es Einsteinium
(c) lmntology.com

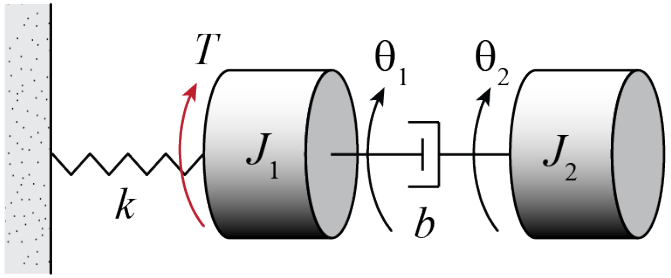

42 Mo Molybdenum
123 D Dilithium
125 E Etherium
3 Li Lithium
7 N Nitrogen
119 G Gundanium
(c) lmntology.com

Modeling Mechanical Systems

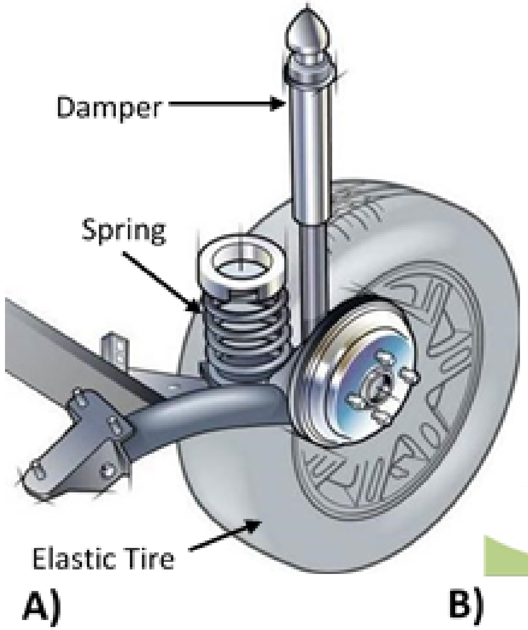
Mechanical systems consist of three basic types of elements:

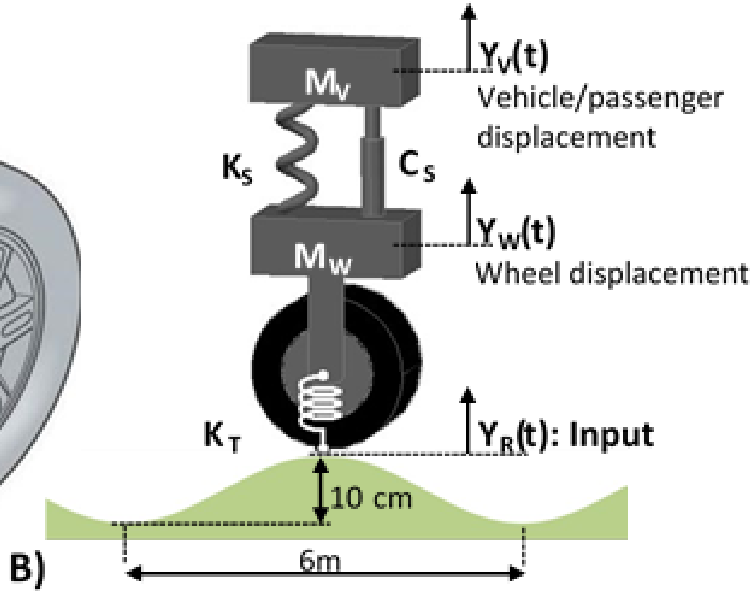
1. Inertia elements
2. Spring elements
3. Damper elements





Vehicle suspension – Mass-spring-damper





Madhur Behl - madhur.behl@virginia.edu

Inertia elements

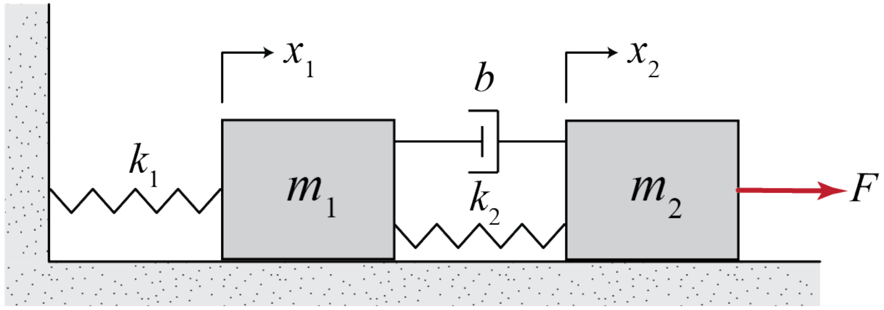
- Example: any mass in the system, or moment of inertia.

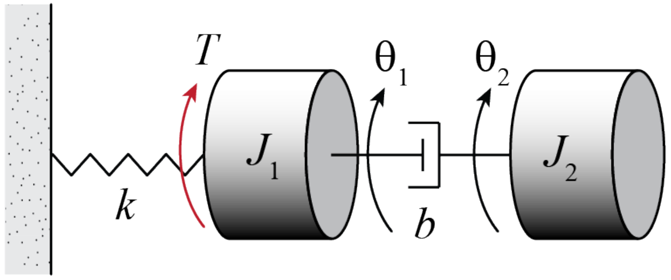
- Each inertia element with motion needs its own differential equation (Newton's 2nd Law, Euler's 2nd law)

$$\sum F = ma \qquad \sum M = J\alpha$$

- **Inertia elements store kinetic energy**

$$E = \int F v \, dt = \int m \dot{v} v \, dt = \frac{1}{2} m v^2$$



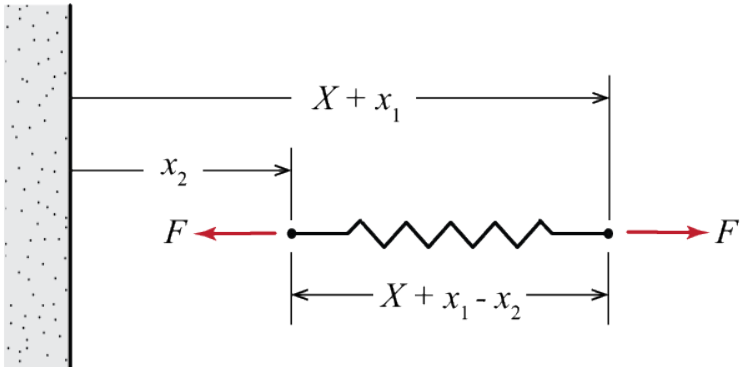


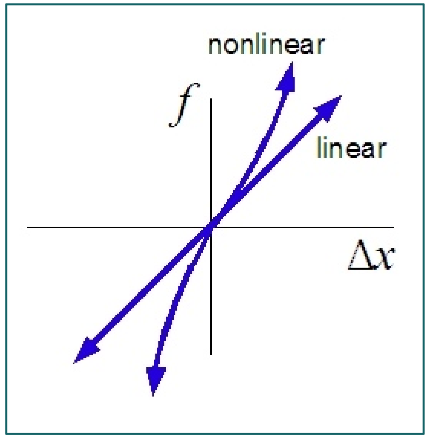
Spring elements

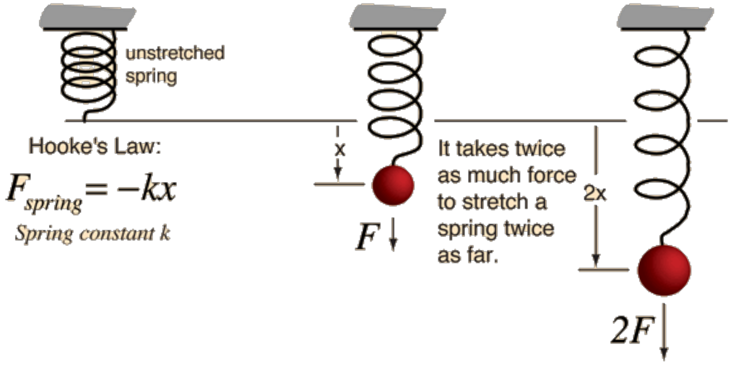
$$F = k(x_1 - x_2)$$

- Force is generated to resist deflection.
- Examples: translational and rotational springs
- **Spring elements store potential energy**

$$E = \int F v \, dt = \int k x \dot{x} \, dt = \frac{1}{2} k x^2$$



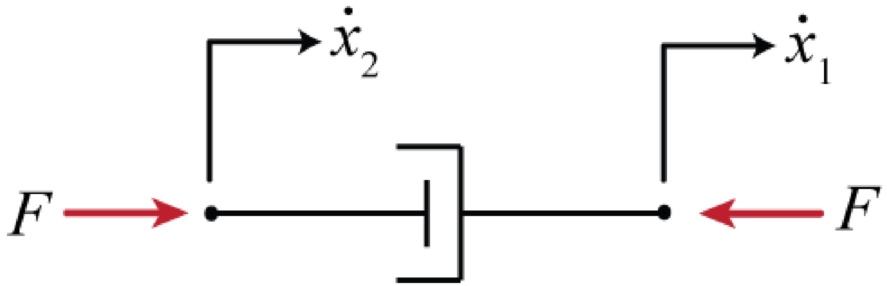




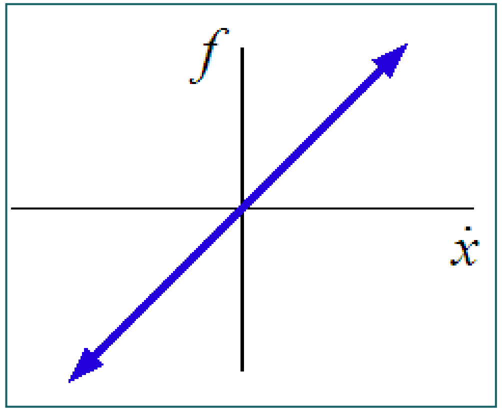
Damper elements

$$F = b(\dot{x}_1 - \dot{x}_2)$$

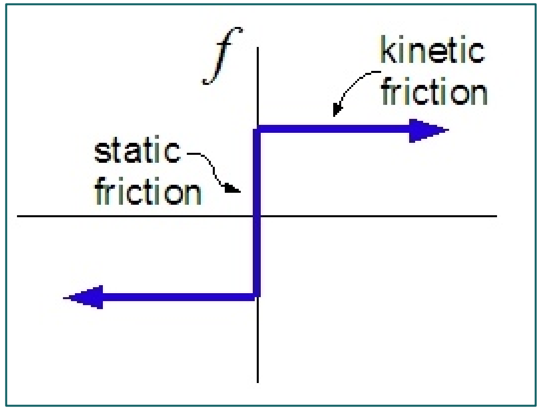
- Force is generated to resist motion.
- Examples: dashpots, friction, wind drag
- **Damper elements dissipate energy**



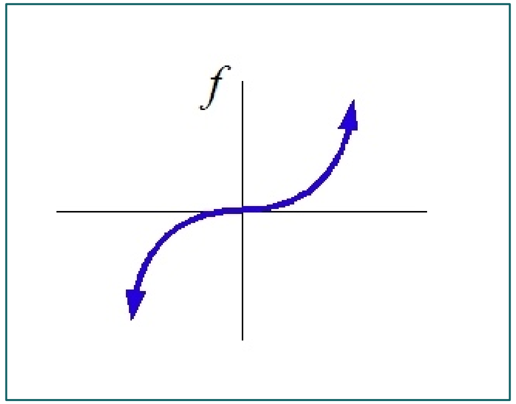
linear damping



friction



drag



How many state variables are required ?

- There is an intuitive way to find state-space models

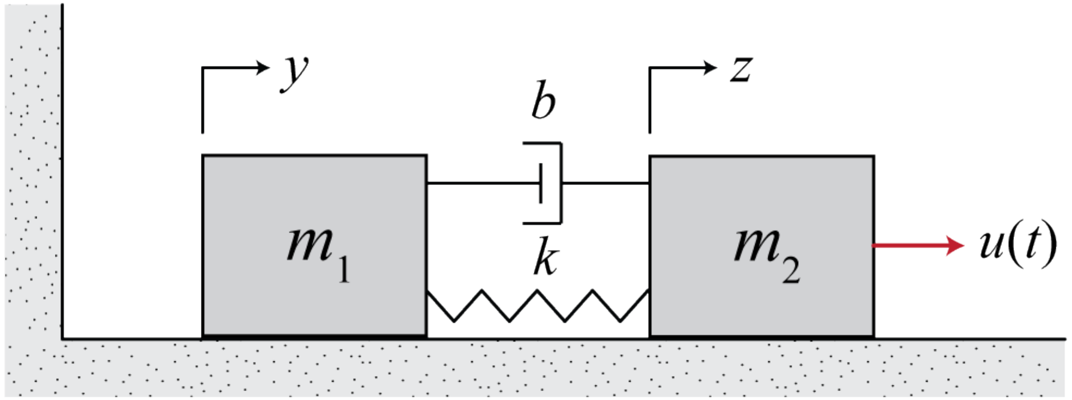
- What initial conditions do I need to capture the system's state?

- Definition: the state of a dynamic system is the set of variables (called state variables) whose knowledge at $t = t_0$ along with knowledge of the inputs for $t \geq t_0$ completely determines the behavior of the system for $t \geq t_0$

- **# of state variables = # of <u>independent</u> energy storage elements**

Madhur Behl - madhur.behl@virginia.edu

Example



Equations of motion

$$m_1\ddot{y} + \mathrm{b}(\dot{y} - \dot{z}) + \mathrm{k}(\mathrm{y} - \mathrm{z}) = 0$$

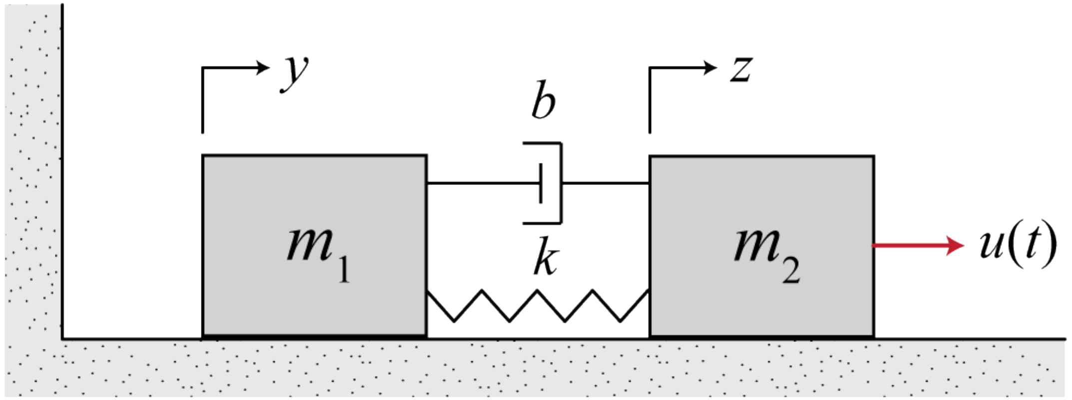
$$m_2\ddot{z} + \mathrm{b}(\dot{z} - \dot{y}) + \mathrm{k}(\mathrm{z} - \mathrm{y}) = \mathrm{u}$$

Choice of state variables

$$x_1 = y, x_2 = \dot{y}$$

$$x_3 = z, x_4 = \dot{z}$$

Example



Equations of motion

$$m_1\dot{x_2} + \mathrm{b}(x_2 - x_4) + \mathrm{k}(x_1 - x_3) = 0$$

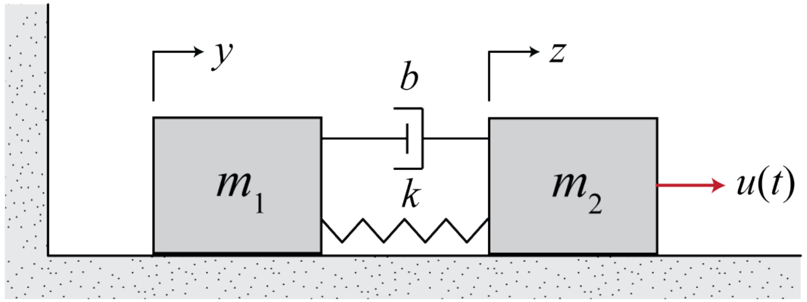
$$m_2\dot{x_4} + \mathrm{b}(x_4 - x_2) + \mathrm{k}(x_3 - x_1) = \mathrm{u}$$

Choice of state variables

$$x_1 = y, x_2 = \dot{y}$$

$$x_3 = z, x_4 = \dot{z}$$

Example



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-\mathrm{b}(x_2 - x_4) - \mathrm{k}(x_1 - x_3)}{m_1}$$

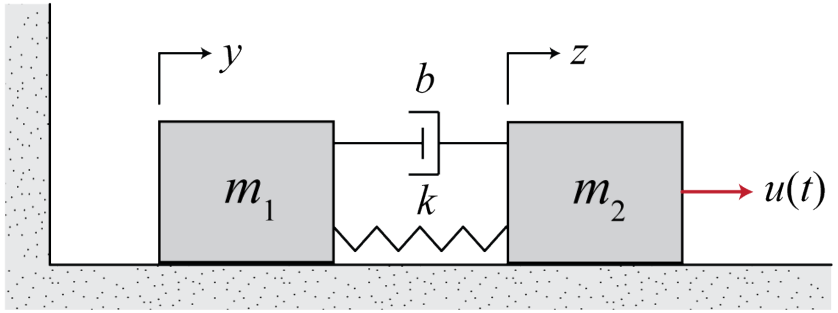
$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{u - \mathrm{b}(x_4 - x_2) - \mathrm{k}(x_3 - x_1)}{m_2}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{m_1} & \frac{-b}{m_1} & \frac{k}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{b}{m_2} & \frac{-k}{m_2} & \frac{-b}{m_2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u$$

Is this the minimum set of states ?

Example

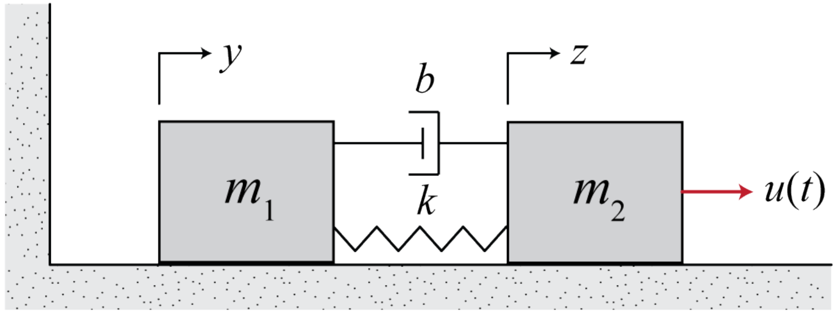


Look at where energy is stored

Energy Storage Element	State Variable
spring (stores elastic PE)	$x_1 = (y - z)$
mass 1 (stores KE)	$x_2 = \dot{y}$
mass 2 (stores KE)	$x_3 = \dot{z}$

damper does not store energy, it dissipates energy

Example



$$m_1\ddot{y} + \mathrm{b}(\dot{y} - \dot{z}) + \mathrm{k}(\mathrm{y} - \mathrm{z}) = 0$$

$$m_2\ddot{z} + \mathrm{b}(\dot{z} - \dot{y}) + \mathrm{k}(\mathrm{z} - \mathrm{y}) = \mathrm{u}$$

$$x_1 = (y - z)$$

$$x_2 = \dot{y}$$

$$x_3 = \dot{z}$$

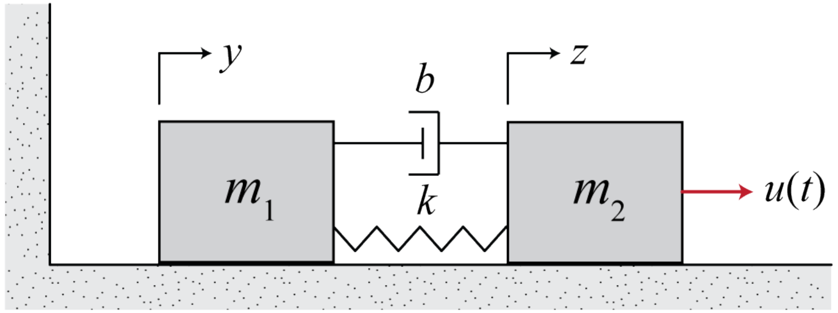
Rewriting in state-space representation

$$\dot{x_1} = x_2 - x_3$$

$$\dot{x_2} = \ddot{y} = \frac{1}{m_1}(-b(x_2 - x_3) - kx_1)$$

$$\dot{x_3} = \ddot{z} = \frac{1}{m_2}(-b(x_3 - x_2) + kx_1 + u)$$

Example



$$\dot{x}_1 = x_2 - x_3$$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m_1}\left(-b(x_2 - x_3) - kx_1\right)$$

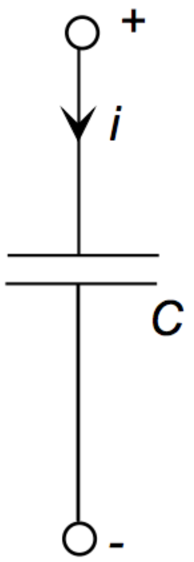
$$\dot{x}_3 = \ddot{z} = \frac{1}{m_2}\left(-b(x_3 - x_2) + kx_1 + u\right)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ \frac{-k}{m_1} & \frac{-b}{m_1} & \frac{b}{m_1} \\ \frac{k}{m_2} & \frac{b}{m_2} & \frac{-b}{m_2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u$$

Modeling electrical systems

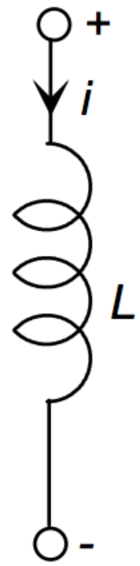
Passive elements

Capacitor
[storage]



$$i = C\frac{dv}{dt}$$

Inductor
[storage]



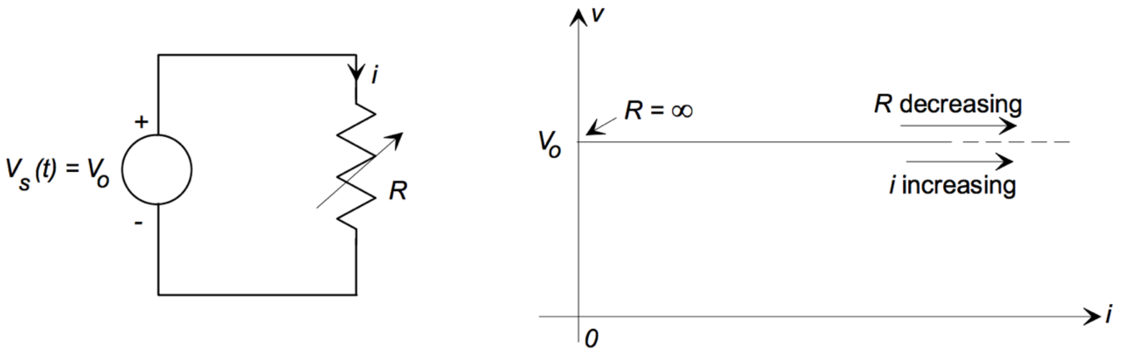
$$v = L\frac{di}{dt}$$

Resistor
[dissipative]

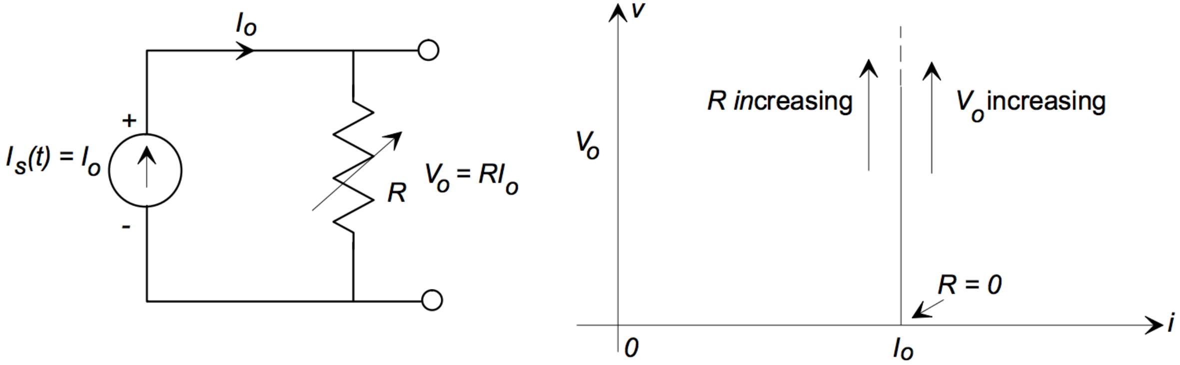
$$v = Ri$$

Active elements

Voltage source



Current source



Mechanical – Electrical equivalency

We recognize a common form to the ODE describing each system and create analogs in the various energy domains, for example:

$$\left\{\begin{array}{l}\text{voltage}\\\text{velocity}\\\text{pressure}\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{current}\\\text{force}\\\text{volume flow rate}\end{array}\right\} \text{ in the } \left\{\begin{array}{l}\text{electrical}\\\text{mechanical}\\\text{fluidic}\end{array}\right\} \text{ domains.}$$

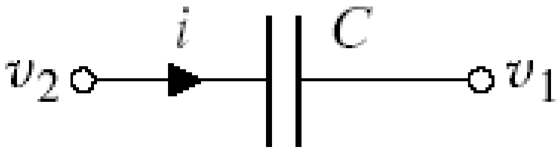
Capacitor - Mass

Describing Equation

Energy

Electrical Capacitance

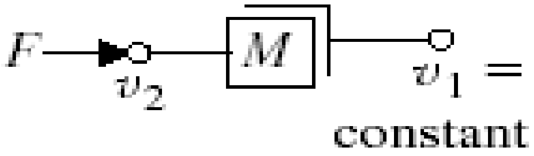
q = CV



$$i = C \cdot \frac{d}{dt} v_{21}$$

$$E = \frac{1}{2} \cdot M \cdot v_{21}^{2}$$

Translational Mass

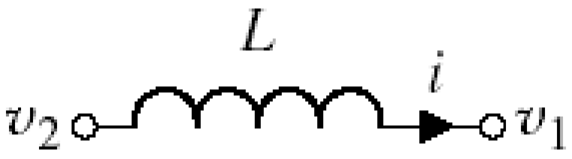


$$F = M \cdot \frac{d}{dt} v_{2}$$

$$E = \frac{1}{2} \cdot M \cdot v_{2}^{2}$$

Inductor - Spring

Electrical Inductance



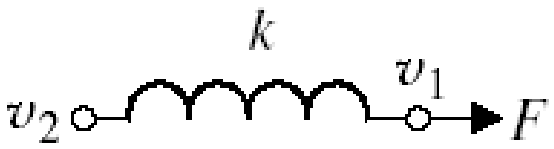
Describing Equation

$$v_{21} = L \cdot \frac{d}{dt} i$$

Energy

$$E = \frac{1}{2} \cdot L \cdot i^2$$

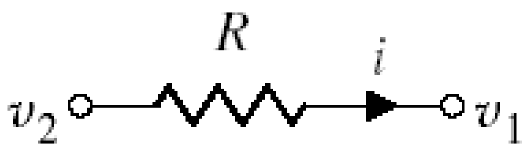
Translational Spring

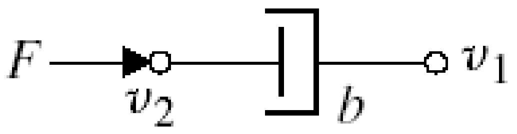


$$v_{21} = \frac{1}{k} \cdot \frac{d}{dt} F$$

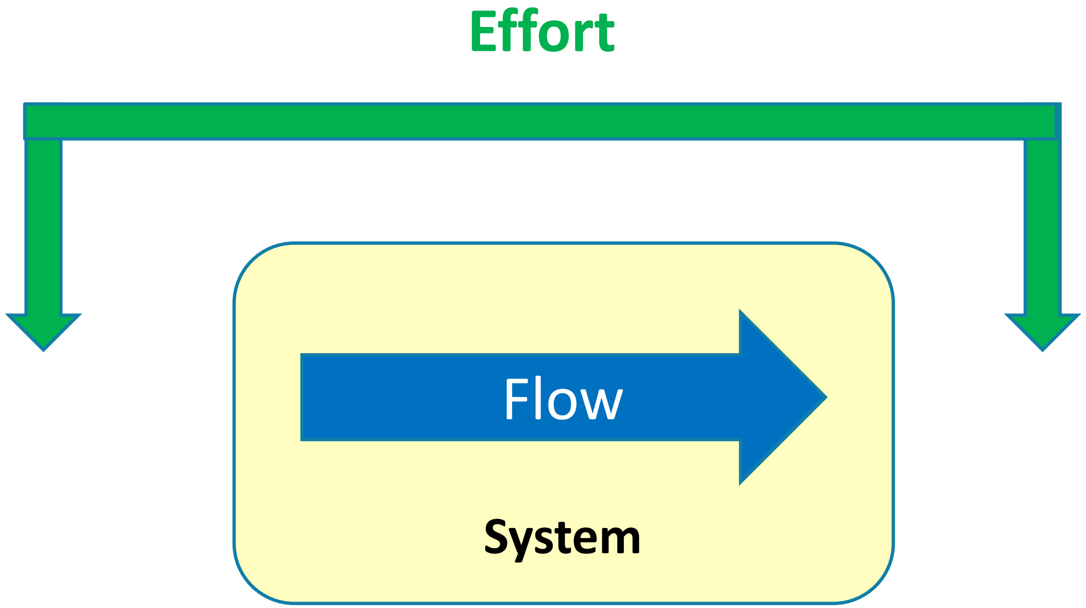
$$E = \frac{1}{2} \cdot \frac{F^2}{k}$$

Resistor - Damper

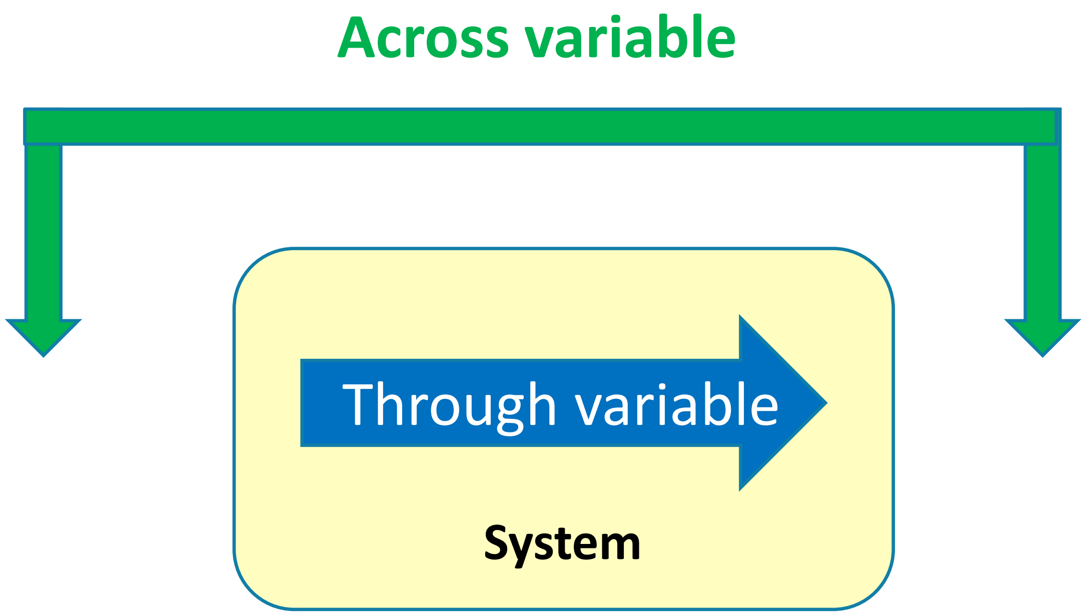
Electrical Resistance	Describing Equation	Energy
	$i = \frac{1}{R} \cdot v_{21}$	$P = \frac{1}{R} \cdot {v_{21}}^2$

Translational Damper		
	$F = b \cdot v_{21}$	$P = b \cdot {v_{21}}^2$

Generalized system representation.



$$Power = Through\ variable \times Across\ variable$$



$$Power = Through\ variable\ \times\ Across\ variable$$

Power is voltage times current

$$P = i \times V$$

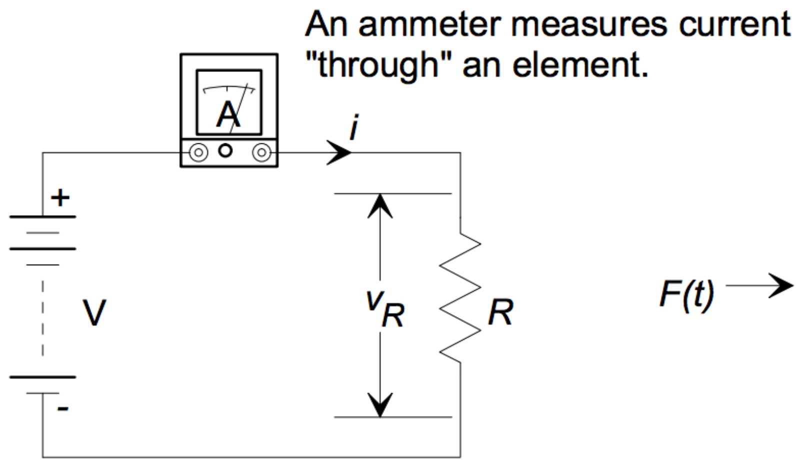
Power is velocity times force

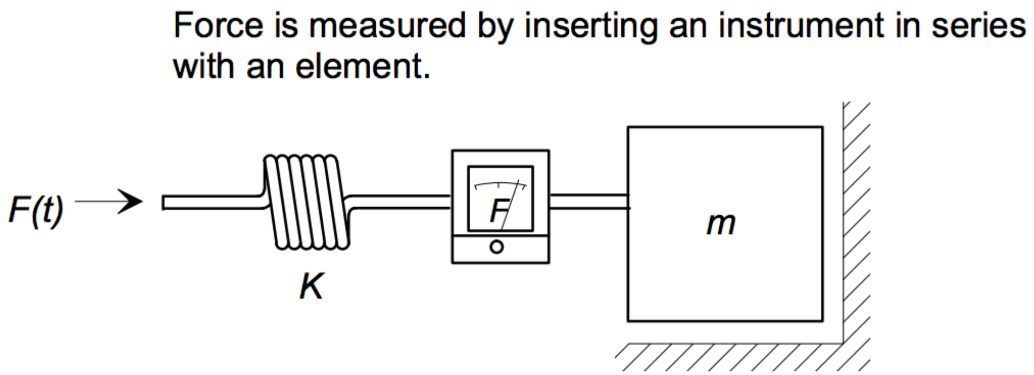
$$P = F \times v$$

$$Power = Through\ variable \times Across\ variable$$

Through variables:

- Variables that are measured through an element.
- Variables sum to zero at the nodes on a graph/circuit/free body diagram.
- Variables that are measured with a gauge connected in series to an element.

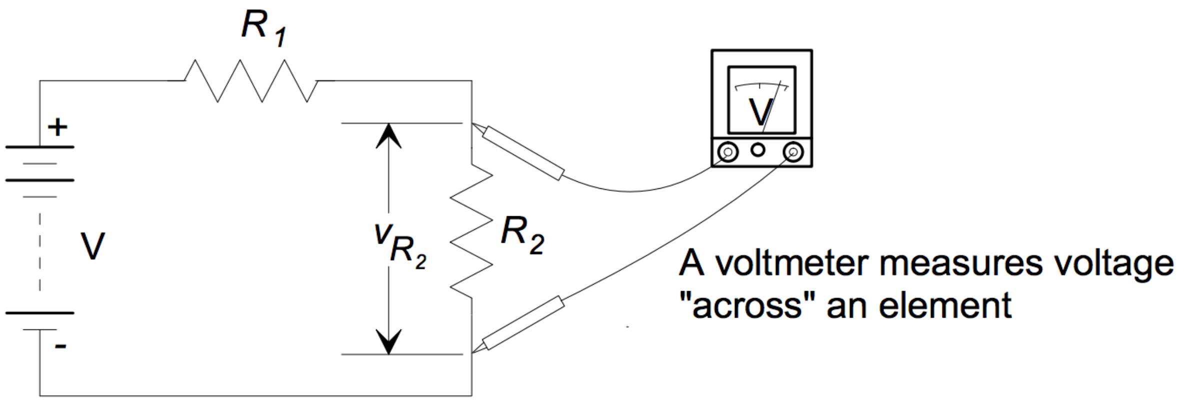




$$Power = Through\ variable \times Across\ variable$$

Across variables:

- Variables that are defined by measuring a difference, or drop, across an element, that is between nodes on a graph (across one or more branches).
- Variables sum to zero around any closed loop on the graph
- Variables that are measured with a gauge connected in parallel to an element.

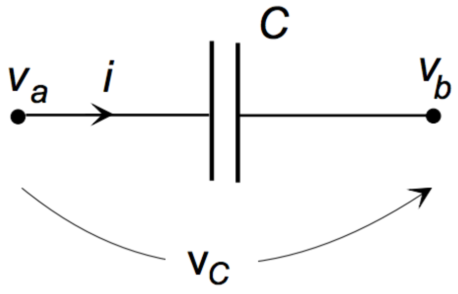


$$Power = Through\ variable \times Across\ variable$$

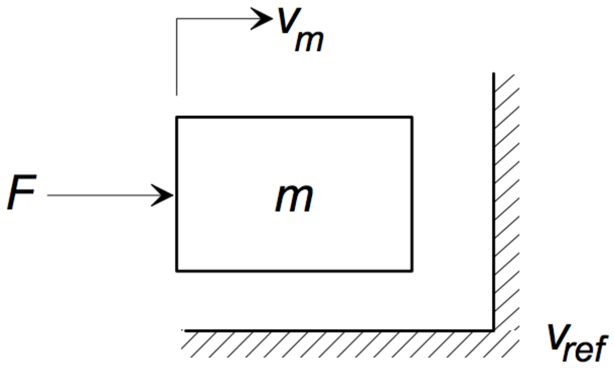
Physical Domain	Across Variable	Through Variable
Electrical	Voltage	Current
Hydraulic	Pressure	Flow rate
Magnetic	Magnetomotive force (mmf)	Flux
Mechanical rotational	Angular velocity	Torque
Mechanical translational	Translational velocity	Force
Gas	Pressure and temperature	Mass flow rate and energy flow rate
Thermal	Temperature	Heat flow
Thermal liquid	Pressure and temperature	Mass flow rate and energy flow rate
Two-phase fluid	Pressure and specific internal energy	Mass flow rate and energy flow rate

Energy storage : A-Type elements

Stored energy is a function of the Across-variable.



$$
\begin{aligned}
i &= C\frac{dv}{dt} \\
E &= \int_{-\infty}^{t} vi\,\mathrm{d}t = \int_{0}^{t} Cv\,\mathrm{d}v \\
&= \frac{1}{2}Cv^2
\end{aligned}
$$



$$
\begin{aligned}
F &= m\frac{dv}{dt} \\
E &= \int_{-\infty}^{t} vF\,\mathrm{d}t = \int_{0}^{t} mv\,\mathrm{d}v \\
&= \frac{1}{2}mv^2
\end{aligned}
$$

Madhur Behl - madhur.behl@virginia.edu

Generalized, Capacitance

generalized across variable

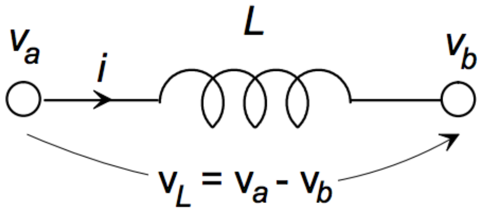
$$f = C\frac{dv}{dt}$$

generalized through variable

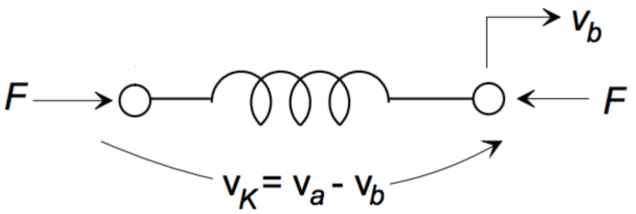
generalized capacitance

Energy storage: T-Type elements

Stored energy is a function of the Through-variable.



$$
\begin{aligned}
v &= L\frac{di}{dt} \\
E &= \int_{-\infty}^{t} vi\,\mathrm{dt} = \int_{0}^{t} Li\,\mathrm{di} \\
&= \frac{1}{2}Li^2
\end{aligned}
$$



$$
\begin{aligned}
v &= \frac{1}{K}\frac{dF}{dt} \\
E &= \int_{-\infty}^{t} vF\,\mathrm{dt} = \frac{1}{K}\int_{0}^{t} F\,\mathrm{d}F \\
&= \frac{1}{2K}F^2
\end{aligned}
$$

Generalized inductance, L

generalized through variable

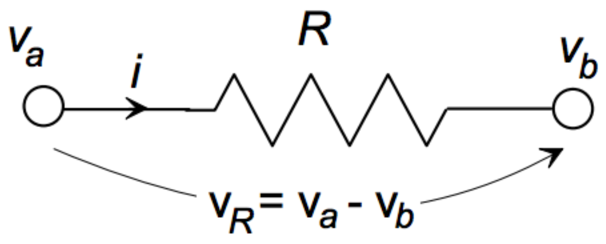
$$v = L\frac{df}{dt}$$

generalized across variable

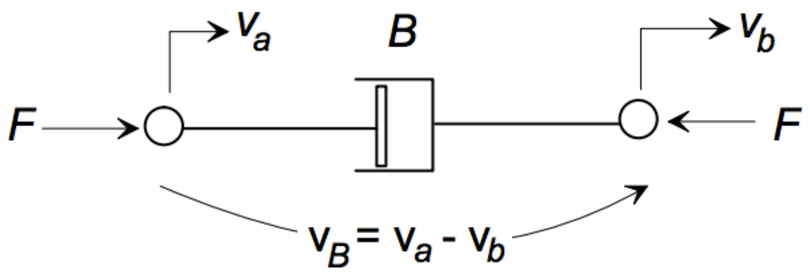
generalized inductance

Dissipative elements : D-Type

Dissipative elements (non-energy storage)

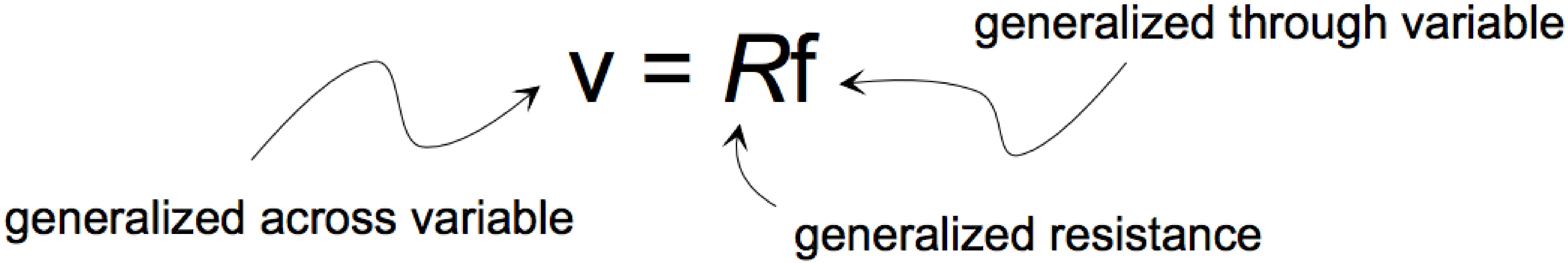


$$
\begin{aligned}
v &= iR \\
P &= vi = i^2R = v^2/R \\
&\geq 0
\end{aligned}
$$



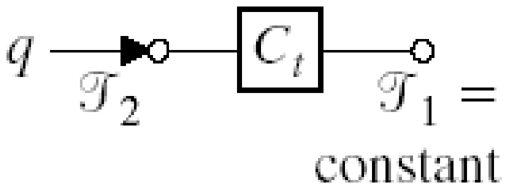
$$
\begin{aligned}
F &= Bv \\
P &= vF = Bv^2 = F^2/B \\
&\geq 0
\end{aligned}
$$

Generalized resistance, R



Cyber-Physical Energy Systems Modeling

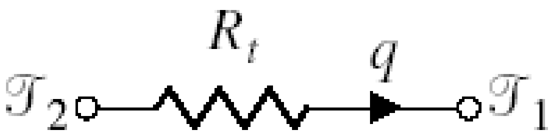
Thermal Capacitance



$$q = C_t \cdot \frac{d}{dt} T_2$$

$$E = C_t \cdot T_2$$

Thermal Resistance



$$q = \frac{1}{R_t} \cdot T_{21}$$

$$P = \frac{1}{R_t} \cdot T_{21}$$

Next lecture..

Learn how to get paid for doing nothing while saving the environment !

....the answer might have to do with drinking tea.