

Linear Dynamical system w. Gaussian noise

$$x_{k+1} = A \cdot x_k + B u_k + \varepsilon$$

$$y_k = C x_k + v$$

\downarrow
 $\in \mathbb{R}^p$ $\underbrace{\hspace{2cm}}_{\in \mathbb{R}^d}$

Given $(\mu_{k|k}, \Sigma_{k|k})$

① Propagation: $\mu_{k+1|k} = A \mu_{k|k} + B u_k$

$$\Sigma_{k+1|k} = A \Sigma_{k|k} A^T + R$$

② Update: $\mu_{k+1|k+1} = \mu_{k+1|k} + K_{k+1} (y_{k+1} - C \mu_{k+1|k})$

$$\Sigma_{k+1|k+1} = (\Sigma_{k+1|k}^{-1} + C^T Q^{-1} C)^{-1}$$

$$K_{k+1} = \Sigma_{k+1|k} C^T (C \Sigma_{k+1|k} C^T + Q)^{-1}$$

In gen:
$$\begin{bmatrix} x_{k+1} = f(x_k, u_k) + \varepsilon_k \\ y_k = g(x_k) + v \end{bmatrix} [D=0]$$

$$P(x_k | y_1, \dots, y_k)$$

Linearization

Let $y = f(x)$ $x \in \mathbb{R}^d$ $y \in \mathbb{R}^p$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1^2 + x_2 x_3 \\ \sin x_2 + \cos x_3 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$$

$p=2$ $d=3$

$$\frac{df(x)}{dx} = \begin{bmatrix} 2x_1 & x_3 & x_2 \\ 0 & \cos x_2 & -\sin x_3 \end{bmatrix} = \text{Jacobian}$$

$J(x)$
 $x=a \Rightarrow J(a)$ $p \times d$

$$y = f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_p(x) \end{bmatrix} \quad \nabla f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial f_p(x)}{\partial x_1} & \dots & \frac{\partial f_p(x)}{\partial x_d} \end{bmatrix}$$

$$f(x) \approx f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

$(x=a)$

$$x \in \mathbb{R}^d \quad f(x) \approx f(a) + \left. \frac{\partial f(x)}{\partial x} \right|_{(x=a)} (x-a) + \text{h.o.t.}$$

$(x=a)$ Taylor Series

$$x \in \mathbb{R}^d \equiv \mathcal{N}(\mu_x, \Sigma_x) \quad \text{if } y = Ax$$

$$y = f(x)$$

$$E(y) = E(Ax)$$

$$= A \cdot \mu_x$$

$$E(y) = ?$$

$$y = f(x) \approx f(\mu_x) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=\mu_x} (x - \mu_x)$$

$$\underbrace{\left. \frac{\partial f(x)}{\partial x} \right|_{x=\mu_x}}_{J(\mu_x)}$$

$$y \approx Jx + f(\mu_x) - J\mu_x$$

$$E(\underbrace{Jx}_{\text{red}} + \underbrace{f(\mu_x)}_{\text{red}} - \underbrace{J\mu_x}_{\text{red}})$$

$$E(y) \approx \cancel{J\mu_x} + f(\mu_x) - \cancel{J\mu_x}$$

$$\approx f(\mu_x)$$

Now, non-lin dyn sys w. Gaussian noise

$$x_{k+1} = f(x_k, u_k) + \varepsilon_k$$

$$y_k = g(x_k) + v_k \quad [D \neq 0]$$

Assume $(\mu_{k|k}, \Sigma_{k|k})$

1. Given

$$\begin{array}{ccc} \circ & \xrightarrow{P} & \circ \\ (\mu_{K|K}, \Sigma_{K|K}) & & (\mu_{K+1|K}, \Sigma_{K+1|K}) \end{array}$$

$$\textcircled{1} \quad X_{K+1} = f(X_K, U_K) + \varepsilon_K$$

$$X_{K+1} \approx \underbrace{f(\mu_{K|K}, U_K)} + \underbrace{\frac{\partial f}{\partial X}}_{\substack{X = \mu_{K|K} \\ A}} (X - \mu_{K|K}) + \varepsilon_K$$

$$A = J(\mu_{K|K})$$

$$X_{K+1} \approx AX + \underbrace{f(\mu_{K|K}, U_K) - A \cdot \mu_{K|K}}_A + \varepsilon_K$$

$$E(X_{K+1}) = \mu_{K+1|K} = f(\mu_{K|K}, U_K)$$

$$\Sigma_{K+1|K} \approx A \Sigma_{K|K} A^T + R$$

$$\textcircled{1} \rightarrow \mu_{K+1|K} = f(\mu_{K|K}, U_K)$$

$$\Sigma_{K+1|K} \approx A \Sigma_{K|K} A^T + R$$

Step 2 : Update

$$Y_{old} : (\mu_{k+1|k}, \Sigma_{k+1|k}) \rightarrow (\mu_{k+1|k+1}, \Sigma_{k+1|k+1})$$

$$y_{k+1} = g(x_{k+1}) + v_{k+1}$$

$$\approx \underbrace{g(\mu_{k+1|k})}_{\substack{x = \mu_{k+1|k} \\ C = J(\mu_{k+1|k})}} + \frac{\partial g}{\partial x} (x_{k+1} - \mu_{k+1|k}) + v_{k+1}$$

$$y_{k+1} \approx C x_{k+1} + g(\mu_{k+1|k}) - C \mu_{k+1|k} + v_{k+1} \leftarrow$$

dummy observation:

$$y'_{k+1} = y_{k+1} - g(\mu_{k+1|k}) + C \cdot \mu_{k+1|k}$$

$$y'_{k+1} \approx C x_{k+1} + v_{k+1} \leftarrow \text{Lin obs of } v_{k+1}$$

$(\mu_{k+1|k}, \Sigma_{k+1|k})$ from step ①

Compute $(\mu_{k+1|k+1}, \Sigma_{k+1|k+1})$

Step 2
update
of KF

$$\mu_{k+1|k+1} = \mu_{k+1|k} + K_{k+1} \left(\underline{y}_{k+1} - C \mu_{k+1|k} \right)$$

$$\mu_{k+1|k+1} = \mu_{k+1|k} + K \left(y_{k+1} - g(\mu_{k+1|k}) \right)$$

$$\Sigma_{k+1|k+1} = \left(\Sigma_{k+1|k}^{-1} + C^T Q^{-1} C \right)^{-1}$$